Toric Varieties

Ref: Fulton, Introduction to Toric Varieties

§ Examples.

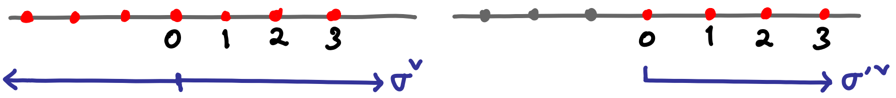
Eg. Cpx torus

affine ⇆ proj.

$$M \quad \binom{\text{say}}{n=1} \quad \underbrace{(\mathbb{C}^\times)^n}_{\text{Spec}\,\underbrace{\mathbb{C}[x,x^{-1}]}_{\mathbb{C}[S_\sigma]}} \subseteq \underbrace{\mathbb{C}^n}_{\text{Spec}\,\underbrace{\mathbb{C}[x]}_{\mathbb{C}[S_{\sigma'}]}} \subseteq \underbrace{\mathbb{CP}^n}_{\mathbb{C}\underset{\mathbb{C}^\times}{\cup}\mathbb{C}}$$

N

M $\quad S_\sigma \subseteq M \simeq \mathbb{Z} \qquad S_{\sigma'} \subseteq M \simeq \mathbb{Z}$

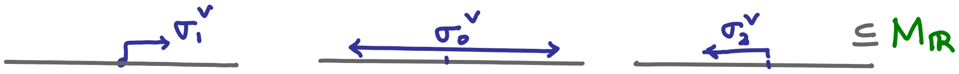


Dually,

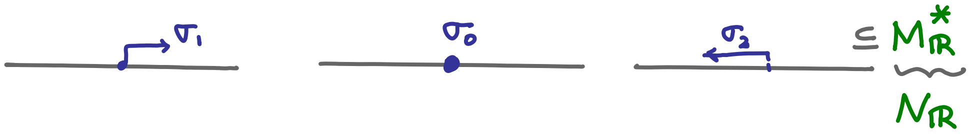
$N \simeq \mathbb{Z} \qquad N$

Non-affine case : gluing.

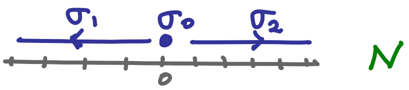
$$\begin{tikzcd} \mathbb{CP}^1 = & \mathbb{C} \arrow[d, dash] & \overset{\cup}{\mathbb{C}^\times} \arrow[d, dash] & \mathbb{C} \arrow[d, dash, "\text{Spec}"] \\ & \mathbb{C}[x] & \mathbb{C}[x^\pm] & \mathbb{C}[x^{-1}] \end{tikzcd}$$



Dually,



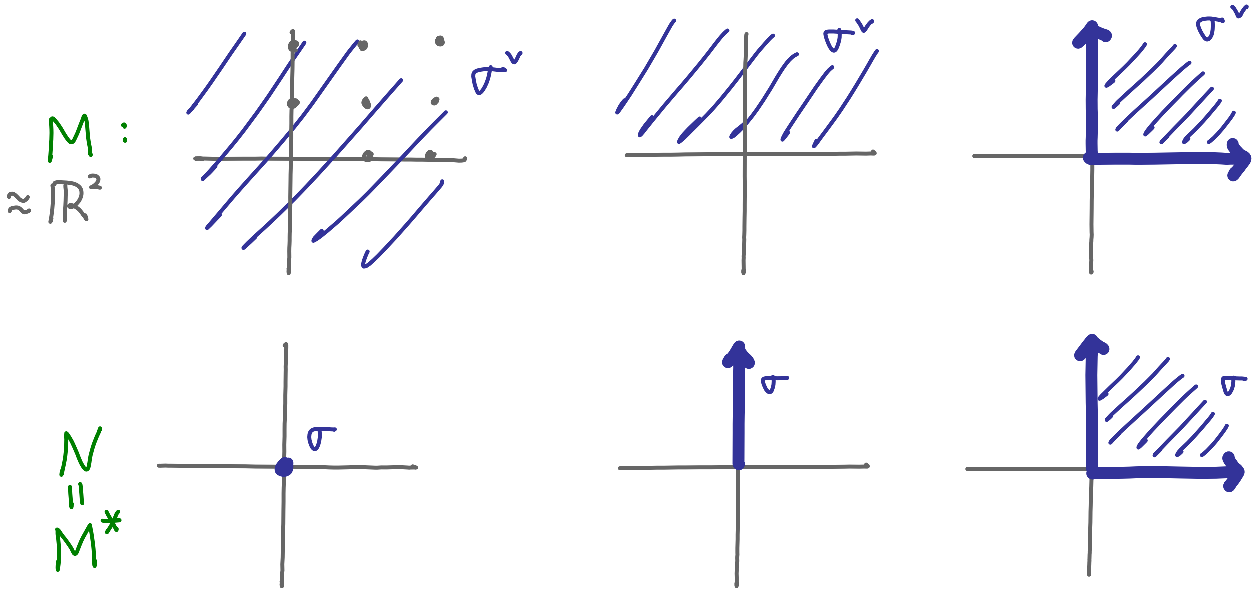
Simply write:



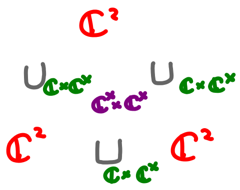
2 dim.

$T = (\mathbb{C}^\times)^2 \subseteq \mathbb{C}^\times \times \mathbb{C} \subseteq \mathbb{C}^2$

$\mathbb{C}[x^\pm, y^\pm]$ $\mathbb{C}[x^\pm, y]$ $\mathbb{C}[x, y]$

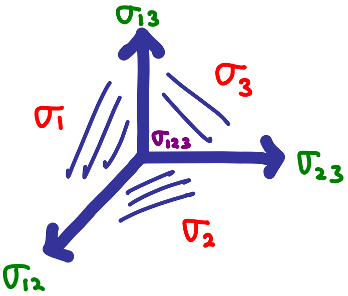


Eg. $\mathbb{CP}^2 =$



$= U_1 \cup U_2 \cup U_3$

$$\begin{cases} U_i \approx \mathbb{C}^2 \\ U_i \cap U_j \approx \mathbb{C} \times \mathbb{C}^\times \\ U_i \cap U_j \cap U_k = (\mathbb{C}^\times)^2 \end{cases}$$



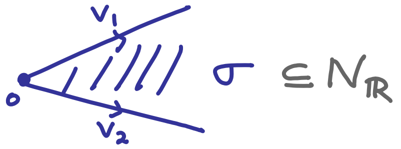
$\subseteq N_{\mathbb{R}}$

§ Affine toric varieties

$$\underset{\text{(fan)}}{\text{Toric Variety}} = \bigcup_{\text{glue}} \underset{(\text{cone} \subseteq N_{\mathbb{R}})}{\text{Affine toric varieties.}}$$

$\sigma \subseteq N_{\mathbb{Q}} \simeq \mathbb{Q}^n$

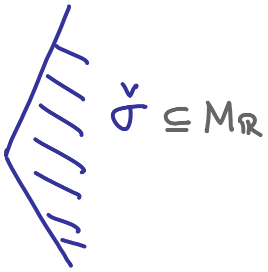
$\sigma = \sum_{j=1}^{s} \mathbb{R}_{\geq 0} v_j$



$\mathbb{Q}$-convex polyhedral cone

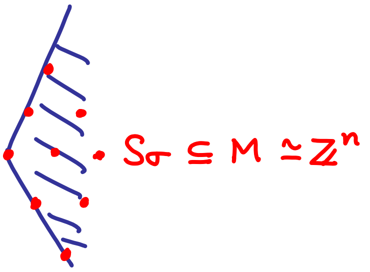
Dual cone $\check{\sigma} \subseteq N_{\mathbb{R}}^{*} = M_{\mathbb{R}}$

$$\check{\sigma} \triangleq \{ u \mid \langle u, v \rangle \geq 0 \ \forall\, v \in \sigma \}$$



$S_\sigma \triangleq \sigma^{\vee} \cap M$

$\sigma / \mathbb{Q} \Rightarrow S_\sigma$: finitely gen. semi-gp.



Def: <u>Affine toric var.</u>

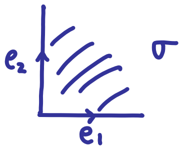
$U_\sigma \triangleq \operatorname{Spec} \mathbb{C}[S_\sigma]$

Say $u = (2,3) \in S_\sigma$

$\chi^u = X_1^2 X_2^3$

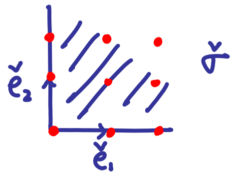
fu. on U_σ

Eg. $\sigma = \sum_{i=1}^{n} \mathbb{R}_{\geq 0} e_i \subseteq N_{\mathbb{R}} \simeq \mathbb{R}^{n*}$



$\Rightarrow \check{\sigma} = \sum_{i=1}^{n} \mathbb{R}_{\geq 0} \check{e}_i \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$

(positive quadric)

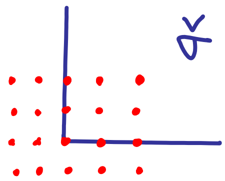


$U_\sigma \simeq \operatorname{Spec} \mathbb{C}[X_1, X_2, \cdots, X_n]$

$\simeq \mathbb{C}^n$

Eg. $\sigma = \{0\} \subseteq N_{\mathbb{R}} \simeq \mathbb{R}^{n*}$, $\check{\sigma} = M_{\mathbb{R}} \simeq \mathbb{R}^n$

$$U_\sigma = \operatorname{Spec} \mathbb{C}[M_{\mathbb{Z}}]$$
$$= \operatorname{Spec} \mathbb{C}[x_1^{\pm}, x_2^{\pm}, \dots, x_n^{\pm}] \simeq (\mathbb{C}^\times)^n$$



In general, $\sigma = \sum_{i=1}^{k} \mathbb{R}_{\geq 0} e_i \Rightarrow U_\sigma \simeq \mathbb{C}^k \times (\mathbb{C}^\times)^{n-k}$.

Remark:

$$\left(\begin{array}{ccccccc} \{0\} & = & \tau & \subseteq & \sigma & = & \sum_{i=1}^{n} \mathbb{R}_{\geq 0} e_i \\ (\mathbb{C}^\times)^n \simeq & & U_\tau & \underset{\text{open}}{\subseteq} & U_\sigma & \simeq & \mathbb{C}^n \end{array}\right)$$

$$\tau \underset{\text{face}}{\subseteq} \sigma \quad \Rightarrow \quad U_\tau \underset{\text{open}}{\subseteq} U_\sigma$$

In particular, $(\mathbb{C}^\times)^n \underset{\text{open}}{\subseteq} U_\sigma$ always.

$$\begin{tikzcd} T_N := (\mathbb{C}^\times)^n \arrow[dr, bend left] \arrow[drr, bend left=40, "\text{Claim}"] & & \\ & (\mathbb{C}^\times)^n \arrow[r, phantom, "\subseteq"] & U_\sigma \end{tikzcd}$$

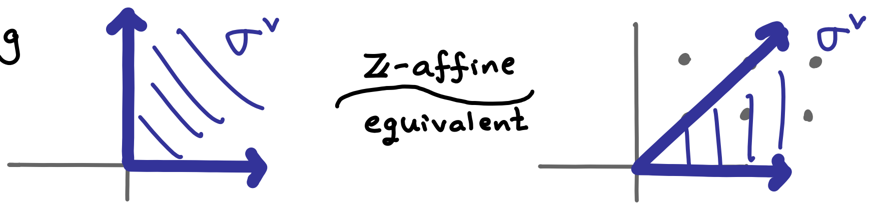
i.e.

$$(\mathbb{C}^\times)^n \times U_\sigma \xrightarrow{?} U_\sigma$$

i.e. (dually)

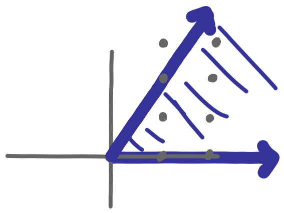
$$\mathbb{C}[S_\sigma] \xrightarrow{?} \mathbb{C}[S_\sigma] \otimes \mathbb{C}[M]$$
$$\chi^u \mapsto \chi^u \otimes \chi^u \quad \checkmark$$

Eg



$$\mathbb{C}^2 = \operatorname{Spec} \mathbb{C}[X,Y] \simeq \operatorname{Spec} \mathbb{C}[X, XY]$$

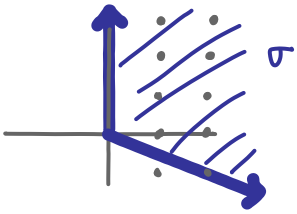
Eg



$\sigma^\vee \subseteq \mathbb{R}^2 = M_{\mathbb{R}}$

$\operatorname{Spec} \mathbb{C}[X, XY, XY^2]$

$\simeq \operatorname{Spec} \mathbb{C}[U,V,W]/UW-V^2$



$\sigma \subseteq N_{\mathbb{R}}$

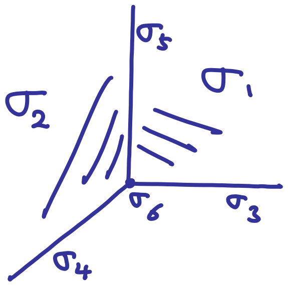
Properties of U_σ :

(i) normal (i.e. $\mathbb{C}[S_\sigma]$ integrally closed)
($\Rightarrow$ codim Sing $\geqslant 2$)

(ii) Cohen-Macauley ($\Rightarrow$ Serre duality ✓)

* (iii) nonsingular / smooth
$\iff$ σ generated by part of a basis for N
$\iff$ $U_\sigma \simeq \mathbb{C}^k \times (\mathbb{C}^\times)^{n-k}$ $k = \dim \sigma$.

§ Toric Varieties $\mathbb{P}_\Delta$



Def: Δ fan

finite collection of cones,

(i) face(cone) is cone

(ii) $\sigma_1 \cap \sigma_2$: face of σ_1 (and σ_2).

$$\mathbb{P}_\Delta = \bigcup_{\sigma\in\Delta} \overbrace{U_\sigma}^{} \xrightarrow{\qquad\qquad} \operatorname{Spec}\mathbb{C}[S_\sigma]$$

gluing: $\tau \underset{\text{face}}{\subset} \sigma \Rightarrow U_\tau \underset{\text{open}}{\subset} U_\sigma$

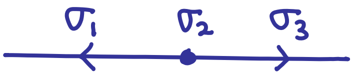
compatible; separated (i.e. Hausdorff)(∵(ii))

Clearly, $(\mathbb{C}^\times)^n \curvearrowright \mathbb{P}_\Delta$.

Eg. $U_\sigma \simeq \mathbb{P}_\Delta$ i.e. affine toric is toric

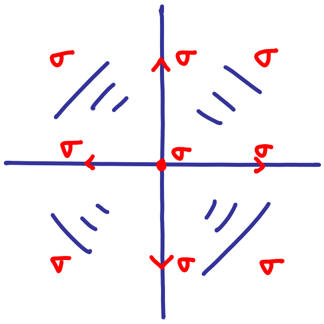
choose $\Delta = \{\tau \mid \tau \underset{\text{face}}{\subseteq} \sigma\}$

Eg. $\mathbb{P}^1$



// $\mathbb{C}$

Eg. $\mathbb{P}^1 \times \mathbb{P}^1$

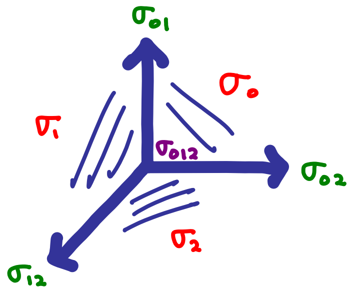


Similar for $\mathbb{P}_{\Delta_1} \times \mathbb{P}_{\Delta_2}$.

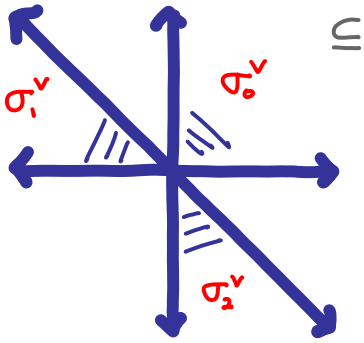
FACT: $\mathbb{P}_\Delta$ compact $\Leftrightarrow$ $|\Delta| = N_\mathbb{R}$

Eg. $$\mathbb{CP}^2 = \begin{array}{ccc} & \mathbb{C}^2 & \\ \cup_{\mathbb{C}\times\mathbb{C}^\times} & {}_{\mathbb{C}^\times\times\mathbb{C}^\times} & \cup_{\mathbb{C}\times\mathbb{C}^\times} \\ \mathbb{C}^2 & \cup_{\mathbb{C}\times\mathbb{C}^\times} & \mathbb{C}^2 \end{array} = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3$$

$$\begin{cases} \mathcal{U}_i \approx \mathbb{C}^2 \\ \mathcal{U}_i \cap \mathcal{U}_j \approx \mathbb{C}\times\mathbb{C}^\times \\ \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k = (\mathbb{C}^\times)^2 \end{cases}$$



$\subseteq N_{\mathbb{R}}$



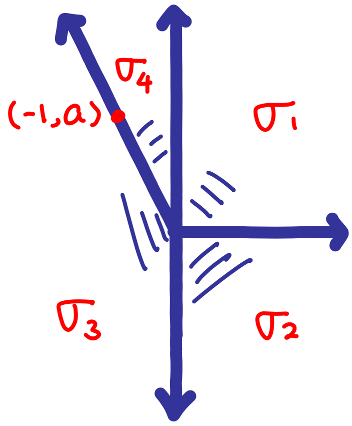
$\subseteq M_{\mathbb{R}}$

$U_{\sigma_0} \simeq \mathbb{C}^2 \qquad X, Y$

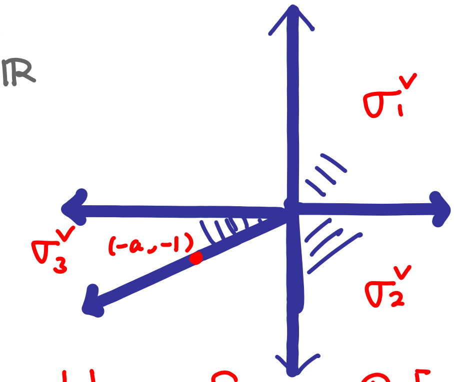
$U_{\sigma_1} \simeq \mathbb{C}^2 \qquad X^{-1}, X^{-1}Y$

$U_{\sigma_2} \simeq \mathbb{C}^2 \qquad Y^{-1}, XY^{-1}$

$\cong \mathbb{P}^2, [T_0, T_1, T_2] \simeq [1, X, Y]$



$\subseteq N_{\mathbb{R}}$



$\subseteq M_{\mathbb{R}}$

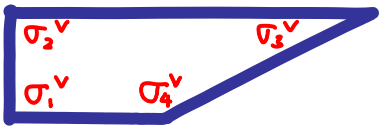
$U_{\sigma_1} = \operatorname{Spec} \mathbb{C}[X, Y]$

$U_{\sigma_2} = \quad$ —"— $\quad X, Y^{-1}$

$U_{\sigma_3} = \quad$ —"— $\quad X^{-1}, X^{-a}Y^{-1}$

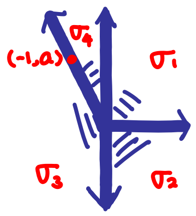
$U_{\sigma_4} = \quad$ —"— $\quad X^{-1}, X^{a}Y$

Up to translation,



polytope

view of view

For 

$\subseteq N_{\mathbb{R}}$, what is $\underbrace{\mathbb{P}_\Delta}_{U_{\sigma_1} \cup U_{\sigma_2} \cup U_{\sigma_3} \cup U_{\sigma_4}}$?

$$\left.\begin{array}{l} U_{\sigma_1} \cup U_{\sigma_2} = \underset{\substack{\operatorname{Spec}\mathbb{C}[X] \\ \times \operatorname{Proj}.\mathbb{C}[Y]}}{\mathbb{C} \times \mathbb{P}^1} \longrightarrow \underset{\mathbb{C}[X]}{\mathbb{C}} \\ \\ U_{\sigma_3} \cup U_{\sigma_4} \simeq \underset{\substack{\operatorname{Spec}\mathbb{C}[X^{-1}] \times \\ \operatorname{Proj}.\mathbb{C}[X^{-a}Y^{-1}]}}{\mathbb{C} \times \mathbb{P}^1} \longrightarrow \underset{\mathbb{C}[X^{-1}]}{\mathbb{C}} \end{array}\right\} \mathbb{CP}^1$$

$\Longrightarrow$ $\mathbb{P}_\Delta \longrightarrow \mathbb{CP}^1$ fiber $\simeq \mathbb{P}^1$

Gluing $Y \rightsquigarrow X^{-a}Y$

$\Longrightarrow$ $\mathbb{P}_\Delta = \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}) = \mathbb{F}_a$ Hirzebruch surface

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Morphism between toric var. :

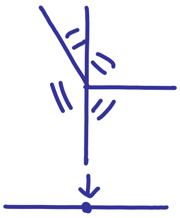
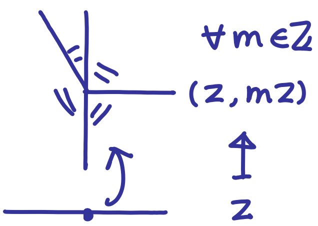
$\Delta' \subseteq N'$ $\Delta \subseteq N$

$\varphi : N' \longrightarrow N$ homo. of lattices

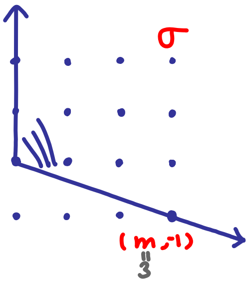
s.t. $\forall$ $\sigma' \in \Delta'$ $\exists$ $\sigma \in \Delta$ s.t. $\varphi(\sigma') \subset \sigma$

$\Longrightarrow$ $U_{\sigma'} \longrightarrow U_\sigma \subseteq \mathbb{P}_\Delta$

$\Longrightarrow$ $\varphi : \mathbb{P}_{\Delta'} \longrightarrow \mathbb{P}_\Delta$

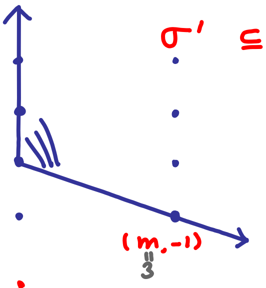
Eg.
$$\begin{tikzcd}\mathbb{P}(\mathcal{O}(a)\oplus\mathcal{O}) \arrow[d] \\ \mathbb{P}^1 \arrow[u, bend right]\end{tikzcd}$$
given by  + 

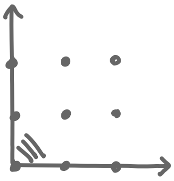
How about fibers?

Eg 

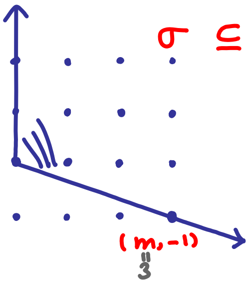
$N \geqslant N'$ sublattice gen. by $(0,1)$, $(m,-1)$

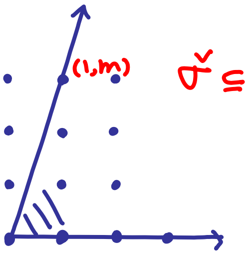
$\sigma' \subseteq N'$



$\cong$ 

$\Longrightarrow \varphi : \underbrace{U_{\sigma'}}_{\mathbb{C}^2} \longrightarrow U_\sigma$ Indeed $U_\sigma \simeq \mathbb{C}^2/\mathbb{Z}_m$

Explicitly, 

$\sigma \subseteq N$ $\rightsquigarrow$ 

$\sigma^\vee \subseteq M$

$\Rightarrow U_\sigma = \operatorname{Spec} \mathbb{C}[S_\sigma]$ with

$$\mathbb{C}[S_\sigma] = \mathbb{C}[X, XY, XY^2, \ldots, XY^m]$$
$$= \mathbb{C}[U^m, U^{m-1}V, \ldots, V^m]$$
$$= \mathbb{C}[U,V]^{\mathbb{Z}_m}$$

$X = U^m$
$Y = V/U$

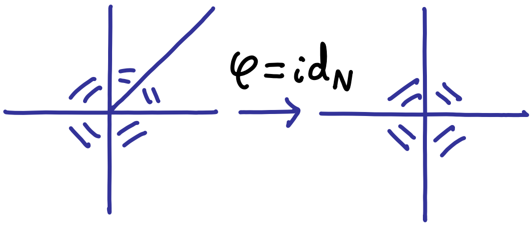
w/ $\mathbb{Z}_m \curvearrowright \mathbb{C}^2$ $\zeta^m = 1$.
$(U,V) \to (\zeta U, \zeta V)$

$\Rightarrow U_\sigma = \mathbb{C}^2/\mathbb{Z}_m$.

(Sublattice ~ finite quotient (by N/N')).

Eg. If Δ' is a refinement of Δ then $\exists$ (birational) morphism,

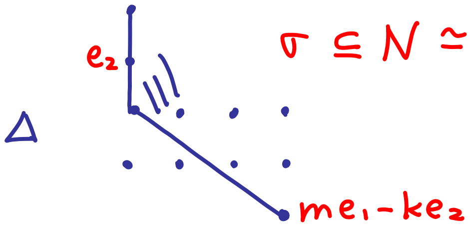
$\varphi : \mathbb{P}_{\Delta'} \longrightarrow \mathbb{P}_{\Delta}$



(birational : $\because$ isom. on T_N)

(Indeed, blow up of a toric subvar.)

Sub-eg. (2 dim)



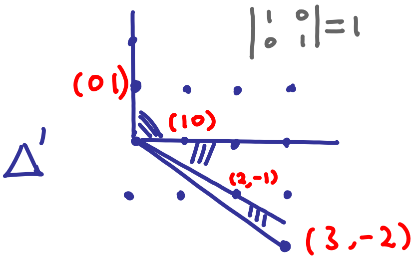
$\sigma \subseteq N \simeq \mathbb{Z}^2$

$(m, k) = 1$, $0 < k < m$

$U_\sigma \simeq \mathbb{C}^2 / \mathbb{Z}_m$, $\mathbb{Z}_m \curvearrowright \mathbb{C}^2$

(via sublattice method). $(\zeta U \ \zeta^k V)$

Refine s.t. each cone is std. ($\Rightarrow$ resol$^{\underline{n}}$.).



$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

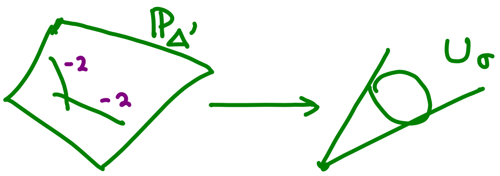
$\begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} = 1$

$\begin{vmatrix} 3 & -2 \\ 2 & -1 \end{vmatrix} = 1$

$(1, 0) + (2, -1) = 2 \ (1 \ 0)$

$(3, -2) + (1, 0) = 2 \ (2, -1)$

$\Longrightarrow$ resol$^{\underline{n}}$ sing. $\mathbb{P}_{\Delta'} \longrightarrow U_\Delta$



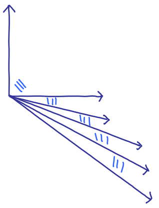
In general, 1° write (continued fraction),

$$\frac{m}{k} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\cdots\cdots - \frac{1}{a_r}}}$$

2° Determine vectors.

$$v_0 = e_2 \quad , \quad v_{r+1} = m e_1 - k e_2$$

$$a_i v_i = v_{i-1} + v_{i+1}$$



$\rightsquigarrow \quad \Delta' \subseteq N_{\mathbb{Z}} \simeq \mathbb{Z}^2$

(each cone span by v_i, v_{i+1} is std. ⇒ smooth.)

$\Longrightarrow$ resol^{n} of sing. $\mathbb{P}_{\Delta'} \longrightarrow U_\sigma \cong \mathbb{C}^2/\mathbb{Z}_m$
$(\zeta u, \zeta^k v)$

$\mathbb{P}_{\Delta'}$

$E_i \simeq \mathbb{P}^1 \quad , \quad E_i^2 = -a_i$

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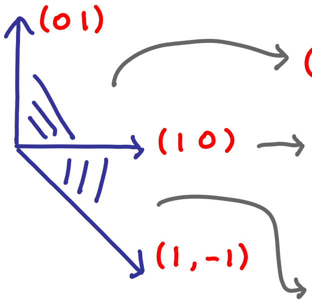
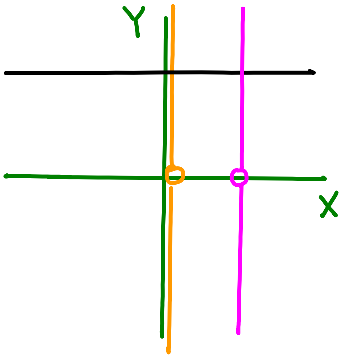
reason:

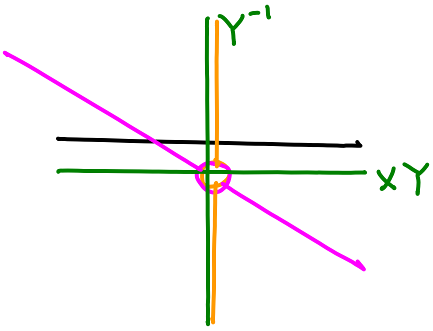


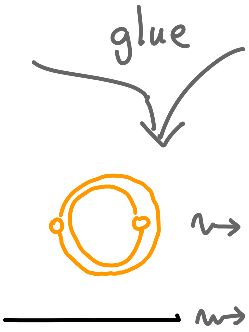
$\mathbb{C}^2 = \operatorname{Spec} \mathbb{C}[X, Y]$

$\mathbb{C} \times \mathbb{C}^\times = \operatorname{Spec} \mathbb{C}[X, Y^{\pm}]$

$\mathbb{C}^2 = \operatorname{Spec} \mathbb{C}[XY, Y^{-1}]$





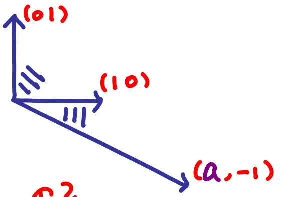


$\rightsquigarrow \mathbb{P}^1$

$\rightsquigarrow$ (fiber) $\mathbb{C}$

$\Longrightarrow \quad \mathbb{C} \longrightarrow \mathbb{P}_\Delta \longrightarrow \mathbb{P}^1 \qquad$ glue : $\quad \mathbb{P}_\Delta = \mathcal{O}_{\mathbb{P}^1}(-1)$

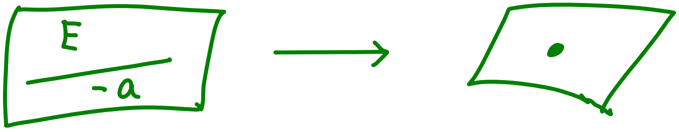
Similarly,

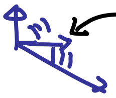


$\begin{vmatrix} a & -1 \\ 1 & 0 \end{vmatrix} = 1$ ✓

Glue $\mathbb{C}^2 \cup \mathbb{C}^2$

$\rightsquigarrow$ $\mathbb{P}_\Delta = \mathcal{O}_{\mathbb{P}^1}(-a)$ $(\because (0,1)+(2,-1)=a(1\ 0))$



( ← give the exceptional div.
(each ray ⟷ toric div. (later)))

Remark: Same procedure resolve any (may not affine) toric surface singularities

Fact: Smooth cpt toric surface ⟺ toric blowup of $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$.

Subeg. $\mathbb{C}^2/\mathbb{Z}_m$ w/ $\mathbb{Z}_m \curvearrowright \mathbb{C}^2$ $\zeta^m = 1$

$(\zeta U, \zeta^{-1} V)$

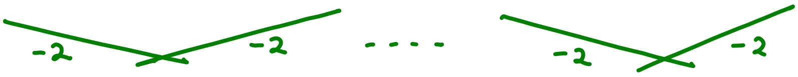
$\Rightarrow \mathbb{Z}_m \leqslant SU(2)$

(Calabi-Yau singularity!)

Note: $\zeta^{-1} = \zeta^{m-1}$, $(m,k) = (k+1,k)$

$$\frac{k+1}{k} = 2 - \cfrac{1}{2 - \cfrac{1}{2 - \cdots\cdots}} \leftarrow \text{k-terms}$$

$\Longrightarrow$ $\mathbb{P}_\Delta$

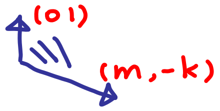


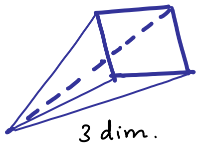
$$\begin{tikzcd} \mathbb{P}_\Delta \arrow[d] \\ \mathbb{C}^2/\mathbb{Z}_{k+1} \simeq U_\sigma = \operatorname{Spec} \dfrac{\mathbb{C}[Y_1, Y_2, Y_3]}{Y_3^{k+1} - Y_1 Y_2} \end{tikzcd}$$

rat$^{\underline{l}}$ double point, type A_k

§ Toric resolution of singularities

First affine case $U_\sigma = \mathrm{Spec}\,\mathbb{C}[S_\sigma]$, $\dim\sigma = n$

- 2 dim. 

Every σ has 2 edges.
- n dim. Every σ has $\geq n$ edges. e.g. 
3 dim.

Def: U_σ simplicial if $\#\text{edges}(\sigma) = n = \dim\sigma$

(similar to 2 dim. situation).

Suppose U_σ simplicial. $\sigma = \sum_{i=1}^{n} \mathbb{R}_{\geq 0} v_i$

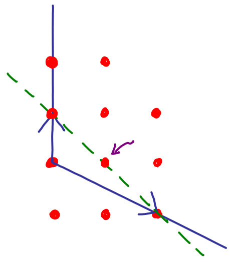
U_σ non-singular $\iff$ $\{v_i\} \sim \{e_i\}$ up to $GL(N_\mathbb{Z})$
i.e. standard

$$\iff \underbrace{[N_\mathbb{Z} : \mathbb{Z}v_1 + \dots + \mathbb{Z}v_n]}_{\text{multi}(\sigma)} = 1$$

Continue U_σ simplicial,

$$\exists \quad v = \sum_{i=1}^{n} t_i v_i \in \sigma \cap N_\mathbb{Z}$$

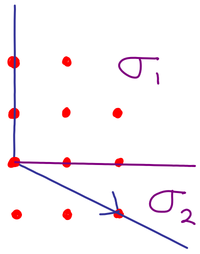
s.t. $0 \leqslant t_i \underset{\neq}{<} 1$



$\frac{1}{2}(0,1) + \frac{1}{2}(2,-1) = (1,0)$

Subdivide using

$v\ , v_1, v_2, \dots, \hat{v}_i, \dots, v_n \rightsquigarrow$ simplicial σ_i

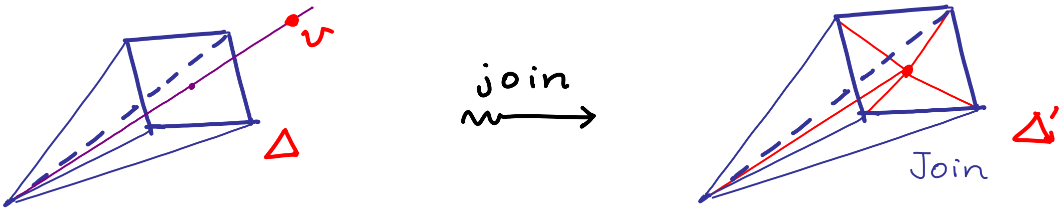


$$\text{multi}(\sigma_i) = t_i \cdot \text{multi}(\sigma)$$
$$\lneq \text{multi}(\sigma)$$

$\rightsquigarrow \mathbb{P}_{\{\sigma_1,\sigma_2\}} \to U_\sigma$

Repeat to get resolution (in simplicial case).

Non-simplicial case:



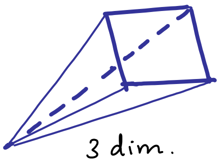
∀ $v \in |\Delta| \cap N$

Make "join" to obtain <u>simplicial</u> refinement Δ'.

$$\underset{\text{simplicial}}{\mathbb{P}_{\Delta'}} \longrightarrow \mathbb{P}_{\Delta}$$

(Can further subdivide to get resol$^{\underline{n}}$.)

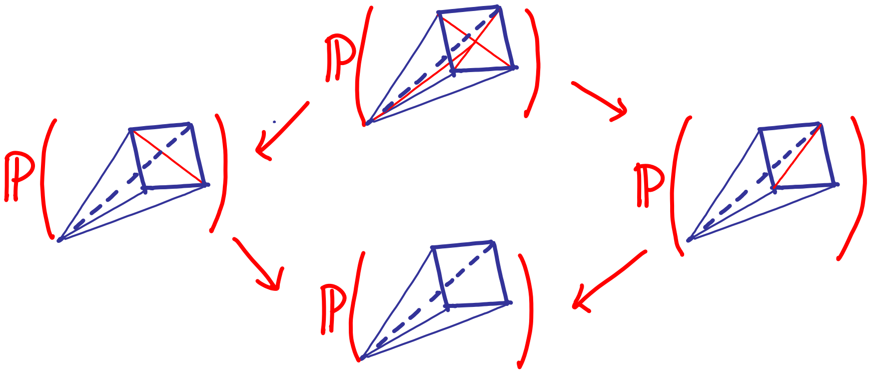
Eg. Cone over quadric,



$\sigma \subset N \simeq \mathbb{Z}^3$

Span by v_1, v_2, v_3, v_4 (spanning N),

s.t. $v_1 + v_3 = v_2 + v_4$.



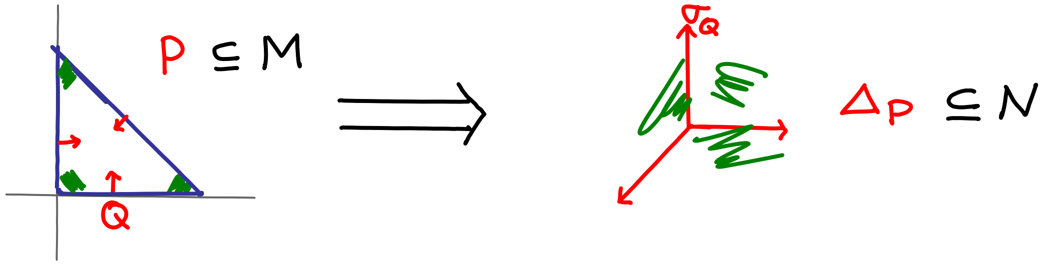
(flop).

Exceptional divs.

$$\begin{tikzcd}
 & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[dl] \arrow[dr] & \\
\mathbb{P}^1 \arrow[dr] & & \mathbb{P}^1 \arrow[dl] \\
 & pt &
\end{tikzcd}$$

Remark: $P \subseteq M_{\mathbb{Q}}$ polytope

$\rightsquigarrow \Delta_P \subseteq N_{\mathbb{Q}}$ fan



$\forall$ face Q of P, define

$$\sigma_Q := \left\{ v \in N_{\mathbb{R}} \;\middle|\; \langle u' - u, v \rangle \geq 0 \ , \begin{array}{l} \forall\, u \in Q \\ \forall\, u' \in P \end{array} \right\}$$

Eg $P = \langle 0, e_1^*, \dots, e_n^* \rangle$ std simplex $\Rightarrow \mathbb{P}_{\Delta_P} = \mathbb{P}^n$

Eg $P = \langle \pm e_1^* \pm e_2^* \pm \dots \pm e_n^* \rangle$ cube $\Rightarrow \mathbb{P}_{\Delta_P} = \mathbb{P}^1 \times \mathbb{P}^1 \times \dots \times \mathbb{P}^1$.

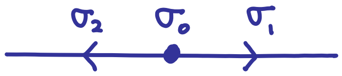
---

§ T_N-orbits.

$$(\mathbb{C}^\times)^n \curvearrowright \mathbb{P}^n_\Delta$$

Eg. $\mathbb{C}^\times \curvearrowright \mathbb{P}^1$

$N \supseteq \Delta$:



$$\mathbb{C}^\times \curvearrowright \overset{\mathbb{C}\ x^{-1}}{\overset{\cong}{U_{\sigma_2}}} \cup \overset{\mathbb{C}\ x}{\overset{\cong}{U_{\sigma_1}}}$$

$U_\sigma \} \mathbb{C}^\times$

$\mathbb{C}^\times$-orbit str: $\underbrace{\{\infty\}}_{O_{\sigma_2}} \cup \underbrace{\mathbb{C}^\times}_{O_{\sigma_0}} \cup \underbrace{0}_{O_{\sigma_1}}$

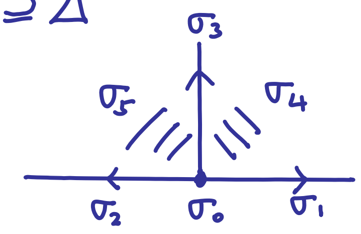
Orbit closure: $\underbrace{\underbrace{\quad}_{V_{\sigma_2}} \qquad\qquad \underbrace{\quad}_{V_{\sigma_1}}}_{V_{\sigma_0}}$

$$V(\tau) = \coprod_{\substack{\sigma \subset \Delta \\ \tau \prec \sigma}} O_\sigma \ , \qquad \tau' \prec \tau \Rightarrow V(\tau) \subseteq V(\tau')$$

$$\dim_{\mathbb{R}} \sigma = \operatorname{codim}_{\mathbb{C}} V_\sigma$$

Eg. $\mathbb{P}^1 \times \mathbb{C}$ $\qquad\qquad \circlearrowleft (\mathbb{C}^\times)^2$

$N \supseteq \Delta$



U_{σ_i} :

$$\begin{array}{ccccc} (\mathbb{C}^\times \cup \infty) \times \mathbb{C} & \supset & \mathbb{C}^\times \times \mathbb{C} & \subset & \mathbb{C} \times \mathbb{C} \\ \cup & & \cup & & \cup \\ (\mathbb{C}^\times \cup \infty) \times \mathbb{C}^\times & \supset & \mathbb{C}^\times \times \mathbb{C}^\times & \subset & \mathbb{C} \times \mathbb{C}^\times \end{array}$$

$(\mathbb{C}^\times)^2$-orbits : O_{σ_i}

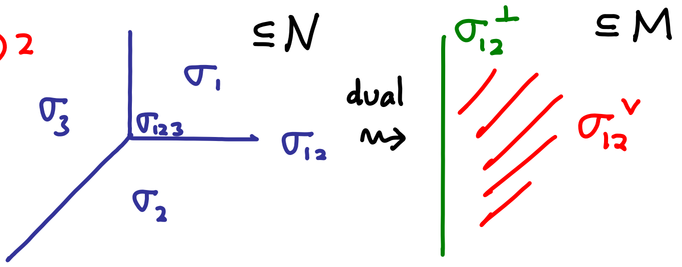
$$\begin{array}{ccccc} \infty \times 0 & \cup & \mathbb{C}^\times \times 0 & \cup & 0 \times 0 \\ \cup & & \cup & & \cup \\ \infty \times \mathbb{C}^\times & \cup & (\mathbb{C}^\times)^2 & \cup & 0 \times \mathbb{C}^\times \end{array}$$

Orbit closure: V_{σ_i}

$$\begin{array}{ccc} \infty \times 0 & \mathbb{P}^1 \times 0 & 0 \times 0 \\ \infty \times \mathbb{C} & \mathbb{P}^1 \times \mathbb{C} & 0 \times \mathbb{C} \end{array}$$

- O_σ closed orbit $\longleftrightarrow$ σ max cone

Eg. $\mathbb{P}^2$



$$\underbrace{U_{\sigma_{12}}}_{\substack{\mathbb{C} \times \mathbb{C}^\times \\ x \quad y}} = \underbrace{\overset{\mathrm{Hom}(\sigma_{12}^\perp,\, \mathbb{C}^\times)}{\underset{eg}{}O_{\sigma_{12}}}}_{\substack{0 \times \mathbb{C}^\times \\ y}} \amalg \underbrace{U_{\sigma_{123}}}_{\mathbb{C}^\times \times \mathbb{C}^\times = O_{\sigma_{123}}}$$

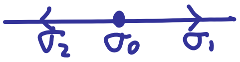
2 orbits ↘ $\mathbb{C} \times \mathbb{C}^\times$

Orbit closure : $V(\sigma_{12}) = \underset{O_{\sigma_{12}}}{(0 \times \mathbb{C}^\times)} \sqcup \underset{O_{\sigma_1}}{0 \times 0} \sqcup \underset{O_{\sigma_2}}{0 \times \infty}$

Note: $\sigma_{12} \prec \sigma_1$ & $\sigma_{12} \prec \sigma_2$.

Construction:

(1) Orbit



$U_{\sigma_1} = \mathbb{C} \ni 0 \in \mathcal{O}_{\sigma_1}$

$$\dim_{\mathbb{C}} \mathcal{O}_\sigma = \operatorname{codim}_{\mathbb{R}}(\sigma \subseteq N) = \dim_{\mathbb{R}}(\sigma^\perp \cap M)$$

$$\mathcal{O}_\sigma = \mathrm{Hom}(\sigma^\perp \cap M, \mathbb{C}^\times)$$

(e.g. $\mathbb{C}^\times = U_{\sigma_0} \simeq \mathcal{O}_{\sigma_0} = \mathrm{Hom}(\overset{\sigma_0^\perp \cap M}{\overbrace{M}}, \mathbb{C}^\times)$)

Write $0 \longrightarrow \underbrace{\sigma^\perp \cap M}_{M(\sigma)} \longrightarrow M \longrightarrow M_\sigma \longrightarrow 0$

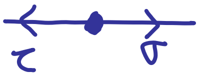
$\overset{\text{dual}}{\Longrightarrow}$ $0 \longrightarrow \underbrace{N_\sigma}_{\text{sublattice of } N \text{ gen. by } \sigma \cap N} \longrightarrow N \longrightarrow N(\sigma) \longrightarrow 0$

$\Longrightarrow \mathcal{O}_\sigma = T_{N(\sigma)} = N(\sigma) \otimes \mathbb{C}^\times \approx (\mathbb{C}^\times)^{n-k}$

$$\begin{tikzcd} \mathcal{O}_\sigma \arrow[r, phantom, "\subset"] \arrow[d, equal] & U_\sigma \arrow[d, equal] \\ \mathrm{Hom}_{\text{semi group}}(\sigma^\perp \cap M, \mathbb{C}) & \mathrm{Hom}_{\text{semi group}}(\sigma^\vee \cap M, \mathbb{C}) \end{tikzcd}$$

defined by extension by zero.

Eg.



$\mathbb{P}^1$

$\sigma^\perp = \{0\}$ $\sigma^\vee \cap M = \{0, 1, 2, \dots\}$

$0 \hookrightarrow U_\sigma = \mathbb{C} = \mathrm{Spec}\,\mathbb{C}[z]$

$\tau^\perp = \{0\}$ $\tau^\vee \cap M = \{\dots -2, -1, 0\}$

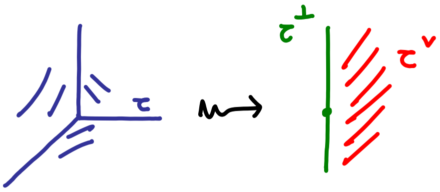
$\infty \hookrightarrow U_\tau = \mathbb{C}^\times \cup \infty = \mathrm{Spec}\,\mathbb{C}[z^{-1}]$

$$\begin{array}{ccc} O_\tau & \subset & U_\tau \\ \| & & \| \\ \underset{\text{semi group}}{\mathrm{Hom}}(\tau^\perp \cap M, \mathbb{C}) & & \underset{\text{semi}}{\mathrm{Hom}}(\tau^\vee \cap M, \mathbb{C}) \end{array}$$

If $\sigma \succ \tau$ (i.e. τ is a face of σ)

$\Rightarrow \sigma^\vee \succ \sigma^\vee \cap \tau^\perp$

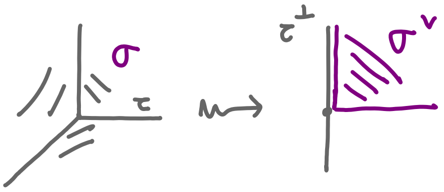
$\mathbb{P}^2$



$$\underbrace{\underbrace{O_\tau}_{\mathrm{Spec}\,[y^{\pm}]}}_{\mathbb{C}^\times} \subset \underbrace{\underbrace{U_{\tau^\vee}}_{\mathrm{Spec}\,\mathbb{C}[x, y^{\pm}]}}_{\mathbb{C}\times\mathbb{C}^\times}$$

(y-axis)\0

[Want closure!]



$$\underbrace{O_\tau}_{\substack{\mathbb{C}^\times \\ \text{y-axis}}} \subset \underbrace{U_{\tau^\vee}}_{\mathbb{C}\times\mathbb{C}^\times}$$

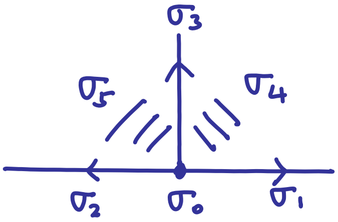
$\sigma \succ \tau$ $\quad U_\sigma(\tau) := \underbrace{\mathrm{Spec}\,\mathbb{C}[\underbrace{\overline{\sigma^\vee} \cap M(\tau)}_{\sigma^\vee \cap \tau^\perp \cap M}]}_{\mathbb{C}[y]} = \mathbb{C}$ y-axis

$$V(\tau) := \bigcup_{\sigma \succ \tau} \underbrace{U_\sigma(\tau)}_{\subseteq U_\sigma}$$

(Note: $U_\tau(\tau) = O_\tau$).

$\Rightarrow V(\tau) \subseteq X(\Delta)$

(2) Orbit closure



$\infty \times 0$	$\mathbb{P}^1 \times 0$	0×0
$\infty \times \mathbb{C}$	$\mathbb{P}^1 \times \mathbb{C}$	$0 \times \mathbb{C}$

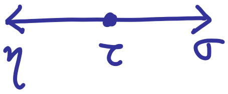
$O_{\sigma_3} \subseteq V(\sigma_3)$ is $\underbrace{\mathbb{C}^\times \times 0}_{N/\langle \sigma_3 \cap N \rangle =} \subseteq \mathbb{P}^1 \times 0$ Want

normal fan!

$\underbrace{Star(\tau)}_{\{\tau + N_\sigma^{\mathbb{R}} / N_\sigma^{\mathbb{R}} \,:\, \sigma \prec \tau\}} \overset{Fan}{\subset} N(\tau)$

(i.e. $\tau = \sigma_3, \sigma_4, \sigma_5$ in eg.)

$V(\sigma) := \mathbb{P}_{Star(\sigma)} \supseteq O_\sigma.$



$O_\sigma = \{0\} = V(\sigma)$, $U_\sigma = \mathbb{C}$

$O_\tau = \mathbb{C}^\times$ $V(\tau) = \mathbb{C}^\times \sqcup 0 \sqcup \infty$, $U_\tau = \mathbb{C}^\times$

$$U_\sigma = \coprod_{\tau \prec \sigma} O_\tau$$

$$V(\sigma) = \coprod_{\gamma \succ \tau} O_\gamma$$

$$\mathbb{P}_\Delta = \coprod_{\underset{\text{cone}}{\sigma} \subset \underset{\text{fan} \subseteq \mathbb{R}^n}{\Delta}} O_\sigma \,, \quad O_\sigma = (\mathbb{C}^\times)^{n-\dim\sigma} \Rightarrow \chi(O_\sigma) = \begin{cases} 1 & \dim\sigma = n \\ 0 & \text{otherwise.} \end{cases}$$

$\Rightarrow$ $\underline{\chi(\mathbb{P}_\Delta)} = \#$ n-dim. cones in $\Delta \geqslant 0$

σ : n-dim cone $\Longleftrightarrow$ $O_\sigma = \{pt\}$ fix pt of $(\mathbb{C}^\times)^n$-act$^{\underline{n}}$.

(every loop in $\mathbb{P}_\Delta$ contract to such pts)

$\exists$ n-dim cone $\Longrightarrow$ $\underline{\pi_1(\mathbb{P}_\Delta)} = 0$

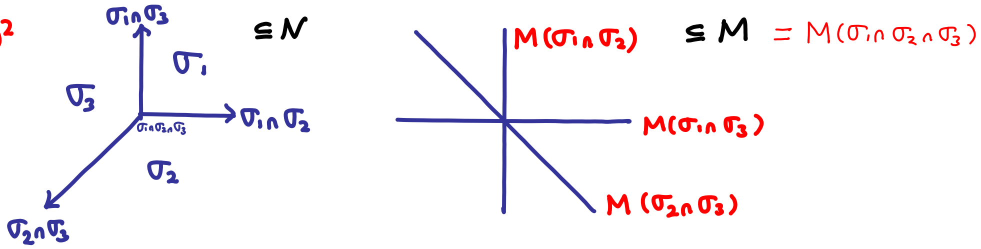
In general, $(N \approx)\, \pi_1(T_N) \twoheadrightarrow \pi_1(\mathbb{P}_\Delta)$.

Other topological property: If Δ complete, then

$$0 \to \underline{H^2(\mathbb{P}_\Delta, \mathbb{Z})} \to \bigoplus_{i<j} M(\underset{\uparrow \text{max cone}}{\sigma_i} \cap \sigma_j) \longrightarrow \bigoplus_{i<j<k} M(\sigma_i \cap \sigma_j \cap \sigma_k)$$

(Hint: Mayer-Vietoris + $H^\bullet(U_\sigma) = \Lambda^* M(\sigma)$).

Eg. $\mathbb{P}^2$



$$0 \to H^2(\mathbb{P}_\Delta, \mathbb{Z}) \to \bigoplus_{i<j} M(\sigma_i \cap \sigma_j) \longrightarrow \bigoplus_{i<j<k} M(\sigma_i \cap \sigma_j \cap \sigma_k)$$

$$0 \to \mathbb{Z} \to \mathbb{Z} + \mathbb{Z} + \mathbb{Z} \longrightarrow \mathbb{Z}^2$$

$$ay + bx + c(x-y) \mapsto (b+c)x + (a-c)y$$

Pf: $\mathbb{P}_\Delta$ open cover by $\underline{\text{contractible}}$ U_{σ_i} $\swarrow$ max cone

Spectral seq. $E_2^{p,q} = \bigoplus_{i_0 < \cdots < i_p} H^q(U_{\sigma_{i_0}} \cap \cdots \cap U_{\sigma_{i_p}}) \Rightarrow H^{p+q}(\mathbb{P}_\Delta)$

$\forall$ cone σ, deformation retract : $U_\sigma \supset O_\sigma \simeq T_{N(\sigma)}$

$\Rightarrow$ $H^q(U_\sigma, \mathbb{Z}) \simeq H^q(T_{N(\sigma)}, \mathbb{Z}) = \Lambda^q M(\sigma)$.

QED.

Remark: $H^2(\mathbb{P}_\Delta, \mathbb{Z}) \simeq \{\text{line bdl}\}/\cong$

$U_{\sigma_i} \times \mathbb{C} \longrightarrow U_{\sigma_i}$

$u \in M(\sigma_i \cap \sigma_j) = (\sigma_i \cap \sigma_j)^\perp \cap M$

$\Rightarrow$ χ^u : nonvanishing function on $U_{\sigma_i} \cap U_{\sigma_j}$

i.e. gluing function.

((u)'s $\longrightarrow 0 \in \oplus M(\sigma_i \cap \sigma_j \cap \sigma_k)$ ← cocycle condition).

$\Rightarrow$ line bundle (with T_N action).

---

§ Divisors. Recall: Say X smooth

- hypersurface $D^{n-1} \subset X^n$

$\longleftrightarrow$ $D \underset{=}{\overset{\text{locally}}{=}} \{ f = 0 \}$ $\quad f : U_\alpha^{\subseteq X} \xrightarrow[\text{holo.}]{} \mathbb{C}$

- $2D_1 - 3D_2 \overset{\text{loc.}}{=} \{ f = 0 \} - \{ f = \infty \}$

f: mero. fu., $f : U_\alpha \xrightarrow[\text{holo.}]{} \mathbb{P}^1$

- Globally, $D = \{ s = 0 \} - \{ s = \infty \} = \operatorname{div}(s)$

$$\begin{tikzcd} \mathbb{C} \arrow[r] & L \arrow[r] & X \arrow[l, bend left, "s"] \end{tikzcd}$$

; $L \cong \mathcal{O}_X(D)$: holo. line bdl. / invertible sheaf.

- D effective if s holo. (i.e. $\{ s = \infty \} = \emptyset$)
- D principal if $L \cong \mathcal{O}_X$ (i.e. $s = f$ global rat^l fu.)

$\{\text{line bdls}\}/\text{isom.} \cong \{\text{div.}\}/\{\text{pr. div.}\} =: \operatorname{Pic}(X)$.

Now. X not necessary smooth.

Weil divisor vs. Cartier divisor

$\overset{\text{finite}}{\sum} a_i \underbrace{V_i}$ $(a_i \in \mathbb{Z})$ hypersurface

$D = \{ (U_\alpha, f_\alpha) : \frac{f_\alpha}{f_\beta} \neq 0 \text{ on } U_\alpha \cap U_\beta \}$

$(f_\alpha$: mero. fu.)

$$\sum_V \mathrm{ord}_V(D) V \longmapsfrom D$$

- X smooth + proj., Weil ≡ Cartier.
- X : normal, then $\{\text{Cartier}\} \subseteq \{\text{Weil}\}$
  (eg. toric)

- D : Cartier ⇒ $\mathcal{O}_X(D)$ line bundle

$$\begin{tikzcd} \mathcal{O}_X(D) \arrow[d] \\ X \arrow[u, bend right, "s"'] \end{tikzcd}$$

$D = \{s = 0\} - \{s = \infty\}$

- $D \geqslant 0$ effective ⇒ $\mathcal{O}_X(-D) = \mathcal{J}_D$
  ideal sheaf

- $f : X \longrightarrow \mathbb{P}^1$ (i.e. $\text{rat}^{\underline{\ell}}$ fu) ⇒ $\mathrm{div}(f)$ (principal) divisor.

$T \curvearrowright \mathbb{P}_\Delta$ T-Weil divisor $\equiv \sum a_i D_i$

$D_i = V(\tau_i)$ w/ τ_i : $\underbrace{\text{1 dim. cone}}_{\text{ray}}$ in Δ

reason: $\mathbb{P}_\Delta \setminus (\cup D_i) = T_N \approx \underbrace{(\mathbb{C}^\times)^n}_{\text{affine}}$ (div ⇒ principal)

$$\underbrace{A_{n-1}(\cup V(\tau_i))}_{\bigoplus_{i=1}^{d} \mathbb{Z} \cdot D_i} \to A_{n-1}(\mathbb{P}_\Delta) \to \underbrace{A_{n-1}(T_N)}_{0}$$

$$\Rightarrow \quad 0 \to M \to \bigoplus_{i=1}^{d} \mathbb{Z} D_i \to A_{n-1}(\mathbb{P}_\Delta) \to 0$$

(Assume Δ non-degen.
i.e. $\dim|\Delta| = n$)

$$\Rightarrow \quad \operatorname{rank}(A_{n-1}(\mathbb{P}_\Delta)) = d - n$$

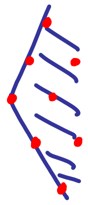
$d = \#$rays in Δ
$= \#$ irred. T-div.

⇒ div always principal.

T-Cartier divisor (affine case)

$T \curvearrowright U_\sigma = \operatorname{Spec} \mathbb{C}[\underbrace{S_\sigma}]$

$\sigma^\vee \cap M \subseteq M \cong \mathbb{Z}^n$



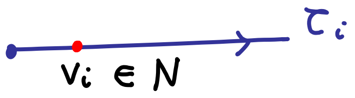
$D = \operatorname{div}(\chi^u) \leftarrow\!\shortmid\; \chi^u \leftarrow\!\shortmid\; u$, $u \in M$

mero. fu. on U_σ

$$\Gamma(U_\sigma, \mathcal{O}(D)) = \mathbb{C}[S_\sigma] \cdot \chi^u \longleftrightarrow u \in \sigma^\vee \cap M$$

fractional ideal.

$$\rightsquigarrow \text{Weil } [\operatorname{div} \chi^u] = \sum_i \langle u, v_i \rangle D_i$$



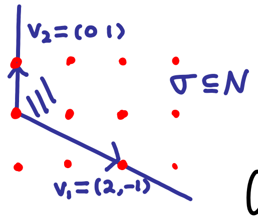
, $D_i = V(\tau_i)$

(i.e. $\langle u, v_i \rangle \overset{?}{=} \operatorname{ord}_{V(\tau_i)}(\operatorname{div} \chi^u)$

Say $M \approx \mathbb{Z} \approx N$

$V(\tau) = 0 \in \mathbb{C}$, $\operatorname{ord}_0(x^u) = u$.)

Eg.



$U_\sigma = \operatorname{Spec} \mathbb{C}[X, XY, XY^2]$

$= \operatorname{Spec} \mathbb{C}[U, V, W]/(V^2 - UW)$

Weil div: $D_1 = V(\tau_1) = \{U = V = 0\}$

$D_2 = V(\tau_2) = \{V = W = 0\}$

T-Cartier div. $u = (p, q) \in M \simeq \mathbb{Z}^2$

$$\operatorname{div} \chi^u = \langle u, \overset{(2,-1)}{v_1} \rangle D_1 + \langle u, \overset{(0,1)}{v_2} \rangle D_2$$

$$= (2p - q) D_1 + q D_2$$

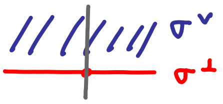
In particular, D_1, D_2 : Not Cartier

But $2D_1, 2D_2$: Cartier.

- $u \in M \rightsquigarrow \operatorname{div}(\chi^u)$ T-Cartier on U_σ

  $u \in \sigma^\perp \cap M = M(\sigma) \Rightarrow \chi^u$ nonvanishing on U_σ ☆

Eg. $\uparrow \sigma \subseteq N = \mathbb{Z}^2$



$U_\sigma = \operatorname{Spec} \mathbb{C}[x^\pm, y] = \underset{(x}{\mathbb{C}^\times} \times \underset{y)}{\mathbb{C}}$, $x^k|_{U_\sigma}$: never zero.

- T-Cartier div on $U_\sigma \longleftrightarrow u \in M/M(\sigma)$

T- Cartier (General Cases)

D : T-Cartier on $\mathbb{P}_\Delta$

$\longleftrightarrow$ $D|_{U_{\sigma_i}}$ $\forall$ $\overset{\text{(max).}}{\text{cone}}$ $\sigma_i \subseteq \Delta$ & compatibility

i.e. $u(\sigma_i) \in M/M(\sigma_i)$ s.t. $u(\sigma_i) - u(\sigma_j) \in M(\sigma_i \cap \sigma_j)$

$$\begin{tikzcd} (u_i) \arrow[d] \\ (u_j - u_i) \end{tikzcd}$$

$$\begin{tikzcd} \{\text{T-Cartier}\} \arrow[d] \arrow[r, phantom, "="] & \mathrm{Ker}\left(\bigoplus_i \frac{M}{M(\sigma_i)} \longrightarrow \bigoplus_{i<j} \frac{M}{M(\sigma_i \cap \sigma_j)} \right) \\ H^2(\mathbb{P}_\Delta ; \mathbb{Z}) \arrow[r, phantom, "\overset{s.s.}{=}"] & \mathrm{Ker}\left(\bigoplus_{i<j} M(\sigma_i \cap \sigma_j) \to \bigoplus_{i<j<k} M(\sigma_i \cap \sigma_j \cap \sigma_k) \right) \end{tikzcd}$$

If $\dim(\max \overset{\sigma_i}{\text{cones}}) = n$, then $M(\sigma_i) = 0$, i.e. $M = \frac{M}{M(\sigma_i)}$,

then $0 \longrightarrow M \longrightarrow \mathrm{Div}_T(\mathbb{P}_\Delta) \longrightarrow H^2(\mathbb{P}_\Delta, \mathbb{Z}) \longrightarrow 0.$

In this case, $H^2(\mathbb{P}_\Delta, \mathbb{Z}) \simeq \mathrm{Pic}(\mathbb{P}_\Delta).$

Remark: Suppose Δ complete, ↻

(i) simplicial

(ii) Weil $\Rightarrow$ $\mathbb{Q}$-Cartier $\mathrm{Pic}(\mathbb{P}_\Delta)_\mathbb{Q} \xrightarrow{\cong} A_{n-1}(\mathbb{P}_\Delta)_\mathbb{Q}$

(iii) $\mathrm{rk}(\mathrm{Pic}(\mathbb{P}_\Delta)) = d - n.$

Pf: Sim licial $\Rightarrow$ orbifold / $\mathbb{Q}$-smooth. So (i) $\Rightarrow$ (ii)

(ii) $\Rightarrow$ (iii) $\because$ $0 \longrightarrow \overset{\mathbb{Z}^n}{\overbrace{M}} \longrightarrow \bigoplus_{i=1}^{d} \mathbb{Z} D_i \longrightarrow A_{n-1}(\mathbb{P}_\Delta) \to 0$

[(iii) $\Rightarrow$ (i)] If NOT, $\exists$ non-simplicial max cone σ w/ generators $v_1, \cdots, v_n, v_{n+1}, \cdots\cdots$

$\Rightarrow$ $\exists$ $a_i \in \mathbb{Z}^+$ s.t. $\{v_1/a_1, \cdots, v_{n+1}/a_{n+1}\} \not\subseteq$ Hyperplane $\subseteq N_\mathbb{R}$

$\Rightarrow$ $\nexists$ $u(\sigma)$ w/ $\langle u(\sigma), v_i \rangle = -k a_i$ $\quad$ $1 \leq i \leq n+1$ $\forall k > 0$.

$\Rightarrow$ $\exists$ Weil & non-$\mathbb{Q}$-Cartier. QED.

D : T-Cartier

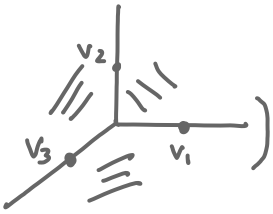
$\rightsquigarrow$ $u(\sigma) \in M/M(\sigma)$ $\forall$ cone $\sigma \subseteq \Delta$

$\rightsquigarrow$ linear $\overset{N_{\mathbb{R}} \supseteq}{|\sigma|} \longrightarrow \mathbb{R}$ (compat.)

$$v \mapsto \underline{-\langle u(\sigma), v\rangle}$$

$\rightsquigarrow$ cts. PL/$\mathbb{Z}$ $\psi_D : |\Delta| \longrightarrow \mathbb{R}$

(recover D as $D = -\sum \psi_D(v_i) D_i$



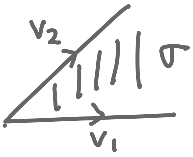
)

- $u \in M$, take $D = \operatorname{div}(\chi^u) \Rightarrow \psi_D = -u$
- $D_1 \equiv D_2 \Longrightarrow \psi_{D_1} - \psi_{D_2}$ linear

---

$H^0(\mathbb{P}_\Delta, \mathcal{O}(D)) = \,?$

- On U_σ, $D = \operatorname{div}(\chi^u)$, $\exists!\, u \in \dfrac{M}{\cancel{M(\sigma)}} \rightarrow$ max cone
- $H^0(U_\sigma, \mathcal{O}(D)) = \left.\underbrace{\mathbb{C}[S_\sigma]}_{\bigoplus_{\substack{w \in M \\ \langle w, v_i\rangle \geq 0 \\ v_i \in \sigma}} \mathbb{C}\chi^w} \cdot \chi^u \right\} \bigoplus_{\substack{u \in M \\ \langle u, v_i\rangle \geq -a_i \\ v_i \in \sigma}} \mathbb{C}\chi^u$

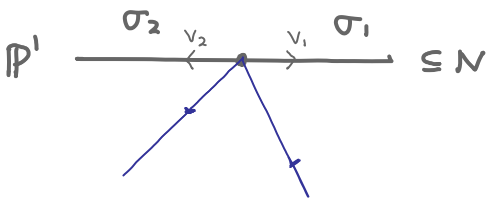


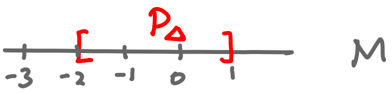
if $D = \sum a_i D_i$

- $H^0(X, \mathcal{O}(D)) = \bigcap_{\sigma \text{ max. cone}} H^0(U_\sigma, \mathcal{O}(D))$

$= \bigoplus_{u \in P_\Delta \cap M} \mathbb{C}\chi^u$ w/ $P_\Delta := \{u \in M_{\mathbb{R}} : \langle u, v_i\rangle \geq -a_i, \forall i\}$

$= \{u \in M_{\mathbb{R}} : u \geq \Psi_D \text{ on } |\Delta|\}$





$$D = 2D_1 + D_2$$
$$= 2(0) + (\infty).$$

$$H^0(\mathcal{O}(D)) = \mathbb{C}\langle x^{-2}, x^{-1}, 1, x \rangle$$

Ex: $|\Delta| = N_\mathbb{R}$ $\Rightarrow$ $\# P_\Delta \cap M$ finite.
i.e. P_Δ complete

Recall $\mathbb{C} \to L \to X$

L ample $\Longleftrightarrow$ $L^{\otimes k}$ very ample
i.e. $\Phi_{|kL|} : X \hookrightarrow \mathbb{P}^N$

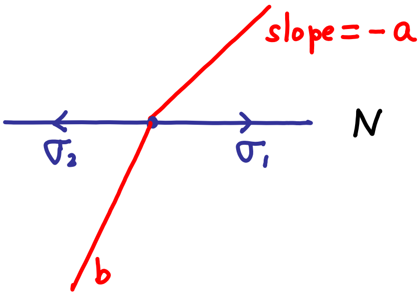
($\Longleftrightarrow$ curvature of $L > 0$)
(Kodaira emb. thm)

L very ample $\Rightarrow$ L globally generated
i.e. $\Phi_{|L|} : X \longrightarrow \mathbb{P}^N$ (Not nec. emb.)

i.e. $\forall\ x \in X$ $\exists\ s \in H^0(X, L)$
s.t. $s(x) \neq 0$.

Eg. $\mathbb{P}^1$ T-Cartier $D = a(0) + b(\infty)$

$\underset{\text{eg.}}{=} -(0) + 2(\infty)$



$$s \in \underbrace{H^0(\mathbb{P}^1, \mathcal{O}(D))}_{H^0(U_{\sigma_1}, \mathcal{O}(D_1)) \cap H^0(U_{\sigma_2}, \mathcal{O}(D_2))}$$

$s|_{U_{\sigma_1}} = \chi^{u(\sigma_1)}$ $\langle u(\sigma_1), v_1 \rangle \geqslant -a$

$= \chi^d$ w/ $d \geqslant -a$

(gen. on U_{σ_1} if $d = -a$)

Similarly, $s|_{U_{\sigma_2}} = \chi^{u(\sigma_2)}$ $\langle u(\sigma_2), \underset{-1}{\underset{\sim}{v_2}} \rangle \geqslant -b$

$= \chi^d$ w/ $d \leqslant b$

Need $a + b \geqslant 0 \Rightarrow \exists d$.

Δ complete (or all max cones have dim n).

$\mathcal{O}(D)$ globally generated ?

Need $\forall$ max. cone σ

$\exists$ $\chi^{u(\sigma)} \in H^0(\mathbb{P}_\Delta, \mathcal{O}(D))$ (i.e. $\exists\, u(\sigma) \in P_D$; i.e. $\langle u(\sigma), v_i \rangle \geqslant -a_i$ $\forall i$)

s.t. $\chi^{u(\sigma)}|_{U_\sigma}$ nonvanishing (i.e. $\forall$ $v_i \in \sigma$, $\langle u(\sigma), v_i \rangle = -a_i$)

$\Longleftrightarrow$ ψ_D is convex.

Similarly, L very ample $\Leftrightarrow$ ψ_D strictly convex.
($\Leftrightarrow$ ample)

- Remark: X sm. proj. toric. D: T-div.

D ample $\Longleftrightarrow$ $D \cdot V(\sigma) > 0$ $\forall (n-1)$ dim cone.

(i.e. $D \cdot (\text{T-curve}) > 0$)

§ Moment Maps (skip).

---

§ Tangent bundles

$X(\Delta)$ smooth toric.

- $K_X^{-1} = \mathcal{O}_X(\sum_{i=1}^{d} D_i)$

Choose any basis $e_1, \dots, e_n$ of M

$$\omega = \frac{d\chi^{e_1}}{\chi^{e_1}} \wedge \frac{d\chi^{e_2}}{\chi^{e_2}} \wedge \dots \wedge \frac{d\chi^{e_n}}{\chi^{e_n}}$$ mero. n-form on $X(\Delta)$

$$= d\log\chi^{e_1} \wedge \dots \wedge d\log\chi^{e_n}.$$

Direct check on each U_σ $\rightsquigarrow$

$\operatorname{div}(\omega) = -\sum D_i$ $\Rightarrow$ ✓

Eg. $\mathbb{CP}^1$, $\mathbb{CP}^2$.

- $$0 \to \Omega^1_X \to \underbrace{\Omega^1_X(\log D)}_{\substack{\wr\| \\ M \otimes_{\mathbb{Z}} \mathcal{O}_X}} \xrightarrow{\text{Res}} \bigoplus_{i=1}^{d} \mathcal{O}_{D_i} \to 0$$

$$\begin{array}{c} d\log\chi^u . \\ \uparrow \\ u \end{array}$$

— Describe $\Omega^k_X(\log D)$, D hypersurface in X (w/ normal crossing)

- loc. free sheaf.
- $\rightsquigarrow$ $H^{\bullet}_{dR}(X \setminus D)$
- above exact sequence.

(see Griffiths + Harris for more details).

Recall Serre duality says $\forall$ vector bundle

$$\mathbb{C}^r \to E \to X$$, we have

$$H^{n-k}(X,\ E^* \otimes \underbrace{\mathcal{O}_X(-\Sigma D_i)}_{K_X}) \cong H^i(X, E)^*$$

for any complete smooth variety X.

Even if toric X $\rightsquigarrow$ Coherent sheaf

$\mathcal{O}_X(-\Sigma D_i)$: shf of rational fu. w/ at least simple zero along D_i's.

(NOT line bundle, but still dualizing sheaf).

Application : Inversion formula: Polytope $P \subseteq \mathbb{Z}^n$

$$f_P(k) := \#(kP) \ , \quad k = 1, 2, 3, \ldots\ldots$$

Claim (i) $f_P(x)$ deg n polynomial

(ii) $f_P(-k) = (-1)^n \, \#(k\mathring{P})$

Pf : $P \rightsquigarrow$ complete toric X w/ D very ample

(refine fan $\rightsquigarrow$ X smooth)

$$\#(kP) = h^0(X, \mathcal{O}(kD)) = \chi(X, \mathcal{O}(kD))$$

↖ poly. (∵ R.R.)

$$\# P^\circ \overset{?}{=} (-1)^n \chi(X, \mathcal{O}(-D))$$

$$0 \to \mathcal{O}_X(-\Sigma D_i) \to \mathcal{O}_X \to \mathcal{O}_X|_{\Sigma D_i} \to 0$$

$\otimes D$ gives

$$0 \to \mathcal{O}_X(D - \Sigma D_i) \to \mathcal{O}_X(D) \to \mathcal{O}_X(D)|_{\Sigma D_i} \to 0$$

long exact seq. ⇒

$$\underbrace{\chi(X, \mathcal{O}(D))}_{\#(P)} = \underbrace{\chi(X, \mathcal{O}(D - \Sigma D_i))}_{\substack{(-1)^n \chi(X, \mathcal{O}(-D)) \\ \text{(Serre duality)}}} + \underbrace{\chi(X, \mathcal{O}(D)|_{\Sigma D_i})}_{\substack{\#(\partial P) \\ \text{(why?)}}}$$

⇒ Done

- D ample ⇒ $D - \Sigma D_i$ gen. by global sections which are χ^u w/ $u \in P^\circ$

$\rightsquigarrow$ $$0 \to \underbrace{H^0(\mathcal{O}(D - \Sigma D_i))}_{\# P^\circ} \to \underbrace{H^0(\mathcal{O}(D))}_{\# P} \to H^0(\mathcal{O}(D)|_{\Sigma D_i}) \to \cancel{H^1} \to \cdots$$

QED.

§ Betti numbers

$X(\Delta)$ toric, proj. & smooth (or simplicial).

Prop: $b_{2k+1} = 0$

= # i dim faces in P.

$$b_{2k} = \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} \overbrace{d_{n-i}}^{\#\,(n-i)\,\text{dim. cones in } \Delta.}$$

Equivalently,

$$P_X(t) := \sum_k b_k t^k = \sum_{k=0}^{n} d_k (t^2-1)^{n-k}$$

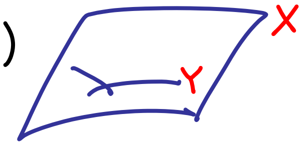
(From ∃ wt. filtration ~ MHS)

X variety (can singular, quasi-proj.)

⇝ poly. $P_X(t)$ virtual Poincaré poly.

s.t. (i) $P_X(t)$ = usual if X smooth & compact.

(ii)



$$P_X = P_Y + P_{X \setminus Y}$$

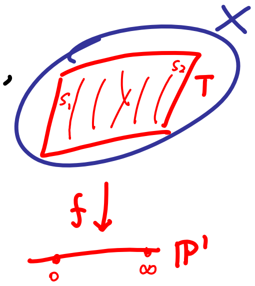
(iii) $P_{X \times Y} = P_X \cdot P_Y$ (also true for fiber bundles).

Eg. $P_{\mathbb{C}^\times} = t^2 - 1$. ⇒ P_{toric} ✓.

§ Intersection Theory

- Modulo rational eq. i.e. $S_1 \sim S_2 \subset X$ dim k

if $\exists\ T \subset X$ dim $k=1$, $f : T \longrightarrow \mathbb{P}^1$, $S_1 = f^{-1}(0)$ & $S_2 = f^{-1}(\infty)$.



⇝ free abelian gp. $A_k(X)$ Chow gp.

- $A_k(X(\Delta))$ generated by $V(\sigma)$'s.
- σ, τ cones in Δ, then

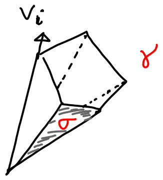
$$V(\sigma) \cap V(\tau) \overset{\text{set}}{=} \begin{cases} V(\gamma) & \text{if } \sigma \,\&\, \tau \text{ span cone } \gamma \\ \phi & \text{otherwise} \end{cases}$$

(easier to see in the polytope picture).

- $(V, D) \mapsto V \cdot D \in A_{k-1}$?

When $V \not\subset D$

$$V(\sigma) \cdot D_i = \sum_{\substack{\gamma:\ n-k+1 \\ \text{dim cone} \\ \text{in } \Delta}} \frac{1}{s_i} V(\gamma)$$



where $\underset{\substack{\updownarrow \\ s_i}}{[v_i]} \in \underset{\substack{\downarrow \cong \\ \mathbb{Z}}}{\underbrace{N_\gamma / N_\sigma}}$

If γ simplicial, then $s_i = \text{multi}(\gamma)/\text{multi}(\sigma)$.

- If $V \subset D$, move V (use rational equiv.)

- Δ simplicial (i.e. $X(\Delta)$ orbifold)

$$A^p(X)_{\mathbb{Q}} = A_{n-p}(X) \otimes \mathbb{Q} \qquad (\to H^{2p}(X))$$

$$A^p(X) \times A^q(X) \longrightarrow A^{p+q}(X)$$ Chow ring. (compat. w/ $\cup$ in $H^\bullet$)

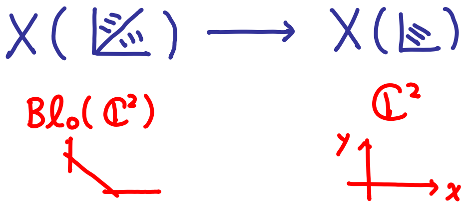
$$\underset{\substack{\text{cone of}\\ \text{dim } p}}{V(\sigma)} \cdot V(\tau) = \frac{\text{multi}(\sigma)\cdot \text{multi}(\tau)}{\text{multi}(\gamma)} V(\gamma)$$

γ: cone of dim $p+q$ span by σ & γ (at most 1).

- General $X(\Delta)$

$f: X(\Delta') \longrightarrow X(\Delta)$ ~ Δ' : refinement to Δ

toric resolution



$\sigma' \subset \Delta'$ s.t. $\underset{\text{same dim}}{\sigma' \subset \sigma} \subset \Delta$

$\Rightarrow$ $f: V(\sigma') \underset{\text{birat.}}{\twoheadrightarrow} V(\sigma)$

and $f_*[V(\sigma')] = [V(\sigma)]$, otherwise $f_*[V(\sigma')] = 0$

$$f_* : (A^\bullet(X(\Delta')), \cdot) \longrightarrow (A^\bullet(X(\Delta)), \cdot)$$

$\rightsquigarrow$ Can determine $\bullet$ for (singular) $X(\Delta)$.

§ Basis for $H^{\bullet}(X(\underbrace{\Delta))}_{\substack{\text{sm. proj.} \\ \text{(or simplicial)}}}$

$$\chi(X) = \#\ n\text{-dim cones } \sigma \text{ in } \Delta$$

$$\| $$

$$\sum b_k(X) \quad (\because\ b_{2l+1} = 0)$$

$\sim$ gen. by $V(\tau)$'s (w/ many relations)

Qu: $\forall$ n-dim cone σ, assoc. τ s.t. $\{V(\tau)\}$ base of $H^{\bullet}(X)$

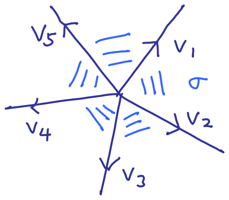
$\sim$ Morse theory (cr. pt. $\rightsquigarrow$ unstable mfd. )

1. Order n-dim cones $\sigma_1, \sigma_2, \ldots\ldots, \sigma_m$.
2. $\tau_1 = 0$, $\tau_2 = \sigma_2 \cap (\bigcup_{k\geqslant 3} \sigma_k)$, ..., $\tau_i = \sigma_k \cap (\bigcup_{k \geqslant i+1} \sigma_k)$, ..., $\tau_m = \sigma_m$.

Need: $\tau_i \subset \sigma_j \Rightarrow i \leqslant j$ (can be arranged, choose $L > 0$)

Thm. $[V(\tau_i)]$ form basis for $H_{\bullet}(X, \mathbb{Z}) \cong A_{\bullet}(X)$
(~Morse). (or $/\mathbb{Q}$).

§ Cohomology Ring



Δ

$V_i \leftrightarrow D_i \subset X$
toric divisor

$X(\Delta)$ sm. proj.

$H^{\bullet}(X)$ (as vector space) span by $V(\sigma)$, $\sigma \underset{\text{cone}}{\subseteq} \Delta$.

- $V(\sigma) = D_{i_1} \bullet D_{i_2} \bullet \ldots\ldots \bullet D_{i_k}$ (can assume no repeat indices).
  if σ span by $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$

$\Rightarrow H^{\bullet}(X)$ (as ring) gen. by D_i's.

Relations: (i) $D_{i_1} \bullet D_{i_2} \bullet \ldots\ldots \bullet D_{i_k} = 0$
if $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ not span a cone in Δ.

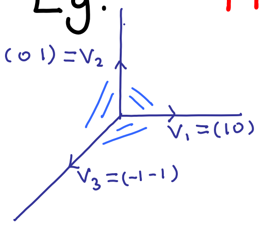
(ii) $\forall\ u \in M$, $\| \ \sum_i \langle u, v_i \rangle D_i = 0$

($\because = \operatorname{div}(\chi^u)$).

Theorem: No other relation!

i.e. $H^\bullet(X(\Delta)) = \mathbb{Z}[D_1, D_2, \dots, D_d] / \langle (i), (ii) \rangle$

Eg. $H^\bullet(\mathbb{P}^2) = \mathbb{Z}[D_1, D_2, D_3] / \sim$



(i) v_1, v_2, v_3 not span a cone $\Rightarrow$ $D_1 \cdot D_2 \cdot D_3 = 0$

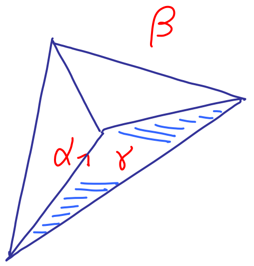
(ii) $\sum \langle u, v_i \rangle D_i = 0$

Take $u = (1\ 0) \Rightarrow$ $D_1 - D_3 = 0$.

Take $u = (0\ 1) \Rightarrow$ $D_2 - D_3 = 0$

$\Rightarrow H^\bullet(\mathbb{P}^2) = \mathbb{Z}[D_1] / D_1^3$.

Key lemma (moving lemma):



$\alpha \underset{\neq}{\leq} \gamma < \beta$ cones in Δ

$\Rightarrow \exists$ relation in $H^\bullet$, $V(\gamma) = \sum m_i V(\gamma_i)$

w/ $\alpha < \gamma_i \not< \beta$, $\dim \gamma_i = \dim \gamma$, $m_i \in \mathbb{Z}$.

i.e. $V(\beta) \subset V(\gamma) \subsetneqq V(\alpha)$

move $V(\gamma)$ away from $V(\beta)$, inside $V(\alpha)$.

Pf: $\underbrace{\underbrace{\underbrace{\underbrace{v_1 \cdots v_p}_{\alpha} \cdots v_k}_{\gamma} \cdots v_q}_{\beta} \cdots v_n}_{\text{basis for } N} \cdots v_d$ (~dual base in M)

Take $u \in M$

$$\begin{cases} \langle u, v_k \rangle = 1 & \\ \langle u, v_i \rangle = 0 & \begin{matrix} 1 \leq i \leq n \\ i \neq k \end{matrix} \end{cases}$$

$\Rightarrow D_k \overset{(ii)}{=} \sum_{j \geq n+1} a_j D_j$

$\Rightarrow V(\gamma) = D_1 \cdot \ldots \cdot D_k = \sum_{j \geq n+1} a_j \overbrace{(D_1 \cdot \ldots \cdot D_{k-1}) \cdot D_j}^{V(\gamma_j) \not\supset} \ V(\beta) = D_1 \cdot \ldots \cdot D_q$.

($\because$ NOT use D_k).

QED.

Pf. of theorem:

Recall $\sigma_1, \sigma_2, \dots\dots, \sigma_m$: n-dim cone, $m = \chi = \mathrm{rk}\, H^\bullet$

$$\rightsquigarrow \underset{\overset{\|}{0}}{\tau_1}, \tau_2, \dots, \overset{\sigma_m}{\overset{\|}{\tau_m}}$$

s.t. $\tau_i \subset \sigma_j \Rightarrow i \leqslant j$

Enough to show $\mathbb{Z}[D_1, \cdots]/(i),(ii)$ span by $V(\tau_i)$'s.

$(\twoheadrightarrow H^\bullet$ w/ $\mathrm{rk} = m)$.

$\forall\ \gamma$, if $\tau_i < \gamma < \sigma_i$, w/ largest i

$i = m \Rightarrow \tau_m = \gamma = \sigma_m \Rightarrow V(\gamma) = V(\tau_m)$ ✓

$\vdots$

i If $\tau_i = \gamma \Rightarrow$ ✓

If $\tau_i \not\leqq \gamma \xRightarrow[\text{lemma}]{} V(\gamma) = \sum\limits_{\tau_i < \gamma_t \not< \sigma_i} m_t V(\gamma_t)$

$\Rightarrow \tau_j < \gamma_t < \sigma_j$ w/ $j \gneqq i \Rightarrow$ ✓ by induction #

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§ Riemann-Roch formula

$$\chi(X, \mathcal{O}(D)) = \int_X ch(\mathcal{O}(D))\, Td(T_X).$$

Recall
$$0 \to \Omega^1_X \to \underbrace{\Omega^1_X(\log(\sum^d D_i))}_{M \otimes_{\mathbb{Z}} \mathcal{O}_X} \to \bigoplus_i^d \mathcal{O}_{D_i} \to 0$$

$$\Rightarrow \quad c(T^*_X) \cdot \prod_{i=1}^d \underbrace{c(\mathcal{O}_{D_i})}_{(1-D_i)^{-1}} = \underbrace{c(\Omega^1_X(\log \Sigma D_i))}_{1}$$

$(\because 0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0)$

$$\Rightarrow \quad c(T^*_X) = \prod_{i=1}^d (1 - D_i)$$

$$\Rightarrow \quad \| \quad c(T_X) = \prod_{i=1}^d (1 + D_i) = \sum_{\sigma \in \Delta} [V(\sigma)]$$

$$ch(T_X) = 1 + c_1(X) + \frac{c_1^2(X) - 2c_2(X)}{2} + \dots\dots$$

$$= \sum_{i=1}^{d} e^{D_i}$$

$$Td(T_X) = 1 + \frac{1}{2}c_1(X) + \frac{c_1^2 + c_2}{12} + \frac{c_1 c_2}{24} + \dots\dots$$

$$= \prod_{i=1}^{d} \frac{D_i}{1 - e^{-D_i}}$$

$$= 1 + \frac{1}{2}\Sigma D_i + \dots\dots + \underbrace{[X]}_{\in H^{2n}(X)}$$ fundamental class

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Given polytope $P \subset M$

$\rightsquigarrow$ complete fan Δ w/

very ample T-Cartier div. D on $X(\Delta)$

$$\forall k \in \mathbb{N},\ H^0(X, \mathcal{O}(kD)) = \bigoplus_{u \in kP} \chi^u$$

$$H^{>0}(\text{ — '' — }) = 0$$

$$\xrightarrow{+RR}\quad \#(kP) = \chi(X, \mathcal{O}(kD))$$

$$= \int_X e^{kD}\, Td(T_X)$$

$$= k^n \frac{D^n}{n!} + k^{n-1} \frac{D^{n-1} \cdot c_1(X)}{(n-1)!2} + \dots\dots$$

For large k,

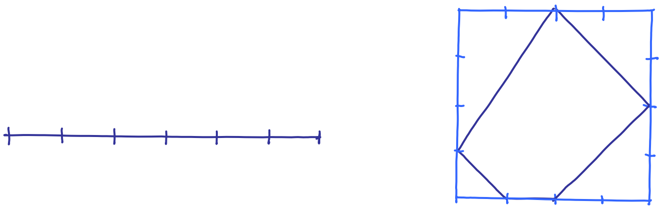
$$\#(kP) \sim \mathrm{Vol}(kP) = k^n \cdot \mathrm{Vol}(P)$$

$$\Rightarrow \quad \mathrm{Vol}(P) = \frac{D^n}{n!}$$

However, RR gives $\#P$, eg.

1 dim. $\#P = \mathrm{Length}(P) + 1$

2dim. $\#P = \mathrm{Area}(P) + \frac{\mathrm{Perimeter}(P)}{2} + 1$



§ Bézout theorem.

Classical thm : $F_1, F_2, \dots, F_n \in \mathbb{C}[X_1, \dots, X_n]$

$\#$ isolated intersection of $\{F_i = 0\}'s \subseteq \mathbb{C}^n \leqslant \prod_{i=1}^{n} (\deg F_i)$.

* * *

$$F : \underbrace{(\mathbb{C}^\times)^n}_{\mathrm{Hom}(M,\mathbb{C}^\times)} \longrightarrow \mathbb{C}$$

$$F = \sum_{u \in M}^{\text{finite}} a_u \chi^u$$

Laurent polynomial.

⇝ Newton polytope $P \subset M$

$$P = \text{Convex Hull}\,\{u \in M : a_u \neq 0\}.$$

Let $Z := \{ z \in (\mathbb{C}^\times)^n : F_1(z) = F_2(z) = \dots = F_n(z) = 0 \}$

Prop. $$\sum_{\substack{z\,:\,\text{isolated} \\ \text{point in } Z}} \underbrace{i(z : F_1, \dots, F_n)}_{\substack{\text{intersection} \\ \text{multiplicity}, \ \in \{1,2,3,\dots\}}} \leqslant n! \underbrace{V(P_1, \cdots, P_n)}_{\substack{\text{mixed} \\ \text{volume.}}}$$

Mixed volume (ANY convex cpt sets, not just polytope)

$$P_1, P_2, \dots, P_n \subset \mathbb{R}^n \rightsquigarrow V(P_1, P_2, \dots, P_n)$$

- $V(P, P, \dots, P) = \mathrm{Vol}(P)$
- $2\,V(P_1, P_2) = \mathrm{Vol}(P_1 + P_2) - \mathrm{Vol}(P_1) - \mathrm{Vol}(P_2)$

- $n!\,V(P_1, \dots, P_n) = E_1 \cdot \dots \cdot E_n$

Here for $P_1 \rightsquigarrow$ complete fan Δ_1 w/ very ample T-Cartier div. E_1 on $X(\Delta_1)$.

$\# P_1 \simeq h^0(X, \mathcal{O}(E_1))$
$u \longleftrightarrow \chi^u$

Refine $\Delta_1 \rightsquigarrow \pi : X(\Delta_1') \xrightarrow{\text{blowup}} X(\Delta_1)$

$$\mathcal{O}(E_1') = \pi^* \mathcal{O}(E_1)$$

E_1' : NOT very ample, still global generated. (same spaces of sections).

Thus $\exists$ Δ compat. w/ ALL $P_1, \cdots, P_n$
& $E_1, \cdots E_n$ globally gen. div.
& $X(\Delta)$ sm. proj. (by refinement).

Now $F_1, \dots, F_n : (\mathbb{C}^\times)^n \longrightarrow \mathbb{C}$

$\rightsquigarrow$ Newton polytopes $P_1, \dots, P_n \subset M \simeq \mathbb{Z}^n$

$\rightsquigarrow$ Δ , $E_1, \dots, E_n$

Lemma: $E_i + \mathrm{div}(F_i)$ effective on $X(\Delta)$

Pf. of Prop:

$$\begin{aligned} RHS &= n! \ V(P_1, \dots, P_n) \\ &= E_1 \cdot \ldots \cdot E_n \\ &= (E_1 + \mathrm{div}(F_1)) \cdot \ldots \cdot (E_n + \mathrm{div}(F_n)) \\ &\geqslant \text{intersection \# in } (\mathbb{C}^\times)^n \in X(\Delta) \end{aligned}$$

But these eff. div. meet $(\mathbb{C}^\times)^n$ on $\bigcap \{F_i = 0\}$.

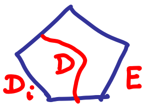
$(\because$ E_i T-div $\Rightarrow$ $\mathrm{Supp}(E_i) \subseteq X(\Delta) \setminus (\mathbb{C}^\times)^n$. $)$

$\geqslant LHS.$ $(\because$ E_i globally gen. $)$.

Proof of lemma: $E + \mathrm{div}(F)$ effective?

i.e. $\forall$ hypersurface $D \subset X(\Delta)$,
order of $E + \mathrm{div}(F)$ along D is $\geqslant 0$.

Case $D \cap (\mathbb{C}^\times)^n \neq \emptyset$



✓ since $F|_{(\mathbb{C}^\times)^n}$ regular

Case $D \subseteq (X \setminus (\mathbb{C}^\times)^n)$, i.e. $D = D_i \leftrightarrow$ Ray $\mathbb{R}_{\geqslant 0} v_i \in \Delta$

$$\mathrm{Ord}_D(\underbrace{F}_{\Sigma a_u \chi^u}) \geqslant \underbrace{\min_{a_u \neq 0}}_{\substack{\min \\ u \in P \cap M}} \underbrace{\mathrm{ord}_D(\chi^u)}_{\langle u, v \rangle} = \underbrace{\psi_P(v)}_{-\mathrm{ord}_D(E)}$$

i.e. $\mathrm{ord}_D(E + \mathrm{div}(F)) \geqslant 0$ QED.

§ Stanley's theorem (skip).

Complete charactenization of all possible # of faces of convex simplicial polytope.

(Pf: Hand Lefschetz thm. for $X(\Delta)$).